AlToGeLiS 2024

Problems for Working Sessions

May 3-4, 2024

Geometry and Algebra of Complex Causal Networks

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Exercises.

1. Convince yourself that $H = 1 \to 2 \to 3$ is not structurally identifiable, but it is structurally identifiable if you intervene on X_2.

2. For the SEM

$$\begin{aligned} X_1 &= N_1 \\ X_2 &= N_2 \\ X_3 &= \lambda_{13}X_1 + \lambda_{23}X_2 + N_3 \end{aligned}$$

   for $N_i \sim N(0, \omega_i)$ independent the defining causal DAG is $G = 1 \to 3 \leftarrow 2$.

   (a) Convince yourself that G is structurally identifiable via Markov equivalence.

   (b) Intervene at X_3 to convince yourself that the edges of the v-structure are causally interpretable.

3. Convince yourself that $G = 1 \to 2$ is identifiable when we assume a Gaussian model with nodes 1 and 2 having the same color.

4. Draw the staged tree and LDAG representations of all CStree models on 3 binary variables. For each tree, associate the variables to some events so that the context-specific relations make sense to you.

Open Problems.

1. Enumerate the ways to partition the d-dimensional cube $[0, 1]^d$ into non-overlapping faces of co-dimension at most 3.

2. Give an algebraic proof of the result of Peters and Bühlmann; i.e., that all vertex-colored DAG models are structurally identifiable.

Considerations for Applications.

1. Think of some data sets where there may be clustering of direct causal relations.

2. Think of some data sets that may naturally contain context-specific CI relations.

Emergence of oscillations in a two-layer cascade

Angélica Torres

Problem 1: Some warm up computations consider the following reaction network

$$\mathrm{X}_1 \xrightarrow{\kappa_1} \mathrm{X}_2 \qquad \mathrm{X}_2 + \mathrm{X}_3 \xrightarrow{\kappa_2} \mathrm{X}_1 + \mathrm{X}_4 \qquad \mathrm{X}_4 \xrightarrow{\kappa_3} \mathrm{X}_3. \tag{1}$$

(i) Write the system of ODEs $\dot{x} = f_\kappa(x)$, assuming mass action kinetics

(ii) The system has two conservation laws: $x_1 + x_2 = T_1$ and $x_3 + x_4 = T_2$. Find the BKK bound for the system of polynomial equations

$$\begin{aligned} x_1 + x_2 - T_1 &= 0, \\ \kappa_1 x_1 - \kappa_2 x_2 x_3 &= 0, \\ x_3 + x_4 - T_2 &= 0, \\ \kappa_2 x_2 x_3 - \kappa_3 x_4 &= 0. \end{aligned}$$

Recall that the BKK bound is the mixed volume of the Newton polytopes of each equation.

(iii) Define a new system by substituting the expressions $x_1 = T_1 - x_2$ and $x_3 = T_2 - x_4$ in the equations

$$\begin{aligned} \kappa_1 x_1 - \kappa_2 x_2 x_3 &= 0, \\ \kappa_2 x_2 x_3 - \kappa_3 x_4 &= 0. \end{aligned}$$

Compute the BKK bound for the new system. Which one is better?

Problem 2. Generalization of problem 1

Let $p_1, \ldots, p_n \in \mathbb{R}[x_1, \ldots, x_n]$ a set of multivariate polynomials such that the variety $V(p_1, \ldots, p_n)$ is zero dimensional. Assume, without loss of generality, that the first k polynomials have degree greater than 1, and the last $n-k$ polynomials are linear (This setup models the equilibria of Chemical Reaction Networks in a Stoichiometric compatibility class).

The cardinality of $V(p_1, \ldots, p_n)$ is bounded above by the mixed volume of the Newton polytopes of each p_i. This is known as the BKK bound.

Question: Is there any relation between the BKK bound for the system $p_1, \ldots, p_n$ and the system $\tilde{p_1}, \ldots, \tilde{p_k}$ obtained by finding a parametric solution $\varphi(x_{i_1}, \ldots, x_{i_k})$ of $p_{k+1}, \ldots, p_n$ and defining $\tilde{p_i} = p_i \circ \varphi$?

Fourier quasicrystals

Mario Kummer

Recall that a smooth closed subvariety $X \subseteq (\mathbb{P}^1)^n$ of codimension d is a *strict Lee–Yang variety* if

1. $(z_1, \dots, z_n) \in X \Rightarrow (\bar{z}_1^{-1}, \dots, \bar{z}_n^{-1}) \in X$.
2. If $(z_1, \dots, z_n) \in X$ such that $|z_i| \neq 1$ for some i, then $\mathrm{var}(\log |z|) \geq d$.

Exercise 1 Prove that

$$X = \left\{ \left(\frac{-1+i+t}{-1-i+t}, \ \frac{-i+t}{i+t}, \ \frac{1+i+t}{1-i+t} \right) \mid t \in \mathbb{C} \cup \{\infty\} \right\} \subseteq (\mathbb{P}^1)^3$$

is a strict Lee–Yang variety of codimension 2. What is the multidegree of X?

Exercise 2 Compute some of the Fourier coefficients of X from **??**, i.e. compute

$$\int_{X \cap \mathrm{T}^3} z_1^{k_1} z_2^{k_2} z_3^{k_3} dz_i$$

for some values of $k \in \mathbb{Z}^3$ and i. Prove that this integral is zero when $\mathrm{var}(k) = 2$.

Hint: Use Cauchy's integral formula.

The arguably most popular quasicrystals are Penrose's tilings. We briefly recall their construction. Fix $\gamma_0, \dots, \gamma_4 \in \mathbb{R}$ such that

$$\gamma_0 + \cdots + \gamma_4 = 0.$$

Further let $\zeta \in \mathbb{C}$ be a primitive fifth root of unity and

$$K_j(z) = \lceil \mathrm{Re}(\zeta^{-j} \cdot z) + \gamma_j \rceil$$

for $z \in \mathbb{C}$. The set of vertices of the Penrose tiling with parameters $\gamma_0, \dots, \gamma_4$ is then given by

$$P_\gamma = \{ \sum_{j=0}^{4} K_j(z) \mid z \in \mathbb{C} \}.$$

We consider the matrix

$$L = \frac{2}{5} \cdot \begin{pmatrix} \mathrm{Re}(\zeta^0) & \mathrm{Im}(\zeta^0) \\ \vdots & \vdots \\ \mathrm{Re}(\zeta^4) & \mathrm{Im}(\zeta^4) \end{pmatrix}$$

Problem 3 Compute the Zariski closure of

$$\{ \exp(2\pi i L x) \mid x \in P_\gamma \} \subseteq (\mathbb{C}^\times)^5$$

for your favorite choice of $\gamma_0, \dots, \gamma_4 \in \mathbb{R}$ with $\gamma_0 + \cdots + \gamma_4 = 0$.

Algebra in probabilistic reasoning

Tobias Boege

Problem 1: Gaussian CI implication

Let Σ be the covariance matrix of a regular Gaussian distribution. (Thus Σ is strictly positive definite!) Then $[i \perp\!\!\!\perp j \mid K]$ holds if and only if $|\Sigma_{iK,jK}| = 0$.

1. For a three Gaussian random variables $\mathit{1}, \mathit{2}, \mathit{3}$ show that

$$[\mathit{1} \perp\!\!\!\perp \mathit{2} \mid \mathit{3}] \wedge [\mathit{1} \perp\!\!\!\perp \mathit{3} \mid \mathit{2}] \implies [\mathit{1} \perp\!\!\!\perp \mathit{2}] \wedge [\mathit{1} \perp\!\!\!\perp \mathit{3}].$$

2. For four Gaussian random variables $\mathit{1}, \mathit{2}, \mathit{3}, \mathit{4}$ show that

$$[\mathit{1} \perp\!\!\!\perp \mathit{3}] \wedge [\mathit{1} \perp\!\!\!\perp \mathit{4}] \wedge [\mathit{1} \perp\!\!\!\perp \mathit{4} \mid \mathit{2}, \mathit{3}] \wedge [\mathit{2} \perp\!\!\!\perp \mathit{3} \mid \mathit{1}, \mathit{4}] \implies [\mathit{1} \perp\!\!\!\perp \mathit{4}].$$

(Hint: Primary decomposition.)

Problem 2: Graphical models

The Gaussian graphical model $\mathcal{M}_G$ of a directed acyclic graph $G = (V, E)$ consists of all positive definite $V \times V$ matrices Σ which satisfy

$$[i \perp\!\!\!\perp j \mid \mathrm{pa}(j)] \text{ for all } i < j \text{ such that } i \to j \notin E.$$

Here $<$ is a topological ordering on G and pa denotes the parent set.

1. Show that the two DAGs $\mathit{1} \to \mathit{2} \to \mathit{3}$ and $\mathit{1} \leftarrow \mathit{2} \leftarrow \mathit{3}$ define the same model. What is its dimension? Which dimension did you expect?
2. For any directed acyclic graph G show that if $i \to j$ is an edge, then $[i \perp\!\!\!\perp j \mid \mathrm{pa}(j)]$ does *not* hold for a generic $\Sigma \in \mathcal{M}_G$.
3. What do you think is the right Bayesian network to represent the causal relationships between “Summer”, “Rain barrel is full”, “Ground is wet”, “It rained”, “Sprinkler was on” and “Umbrella is wet”? Compare your models.

Geometric problems related to the Euler Characteristic Transform

Henry Kirveslahti

Introduction

The Euler Characteristic Transform (ECT) [2] is a topological data analysis tool that vectorizes shape data. It can be seen as a topological Radon transform, or as a vectorization of the Persistent Homology Transform (PHT) [4], which is a kind of a Persistent module that fibers over the sphere S^{d-1}. These tools provide a way to digitally analyse non-diffeomorphic shape data, and are in some sense a digitalization of the Kendall Shape Space. These have also been extended to continuous type data, allowing for applications to fMRI-imaging [3].

In theory, the ECT is an injective summary of the original shape data, meaning no information is lost by working with the ECTs instead of the original shapes. However, in practice one always discretizes the transform, and while there are some theorems on how fine of a discretization is needed for an individual shape [2], there is no good answer for what the discretization should look like concretely, and how to choose it for a collection of shapes.

Problem Statement

In the following we define some key concepts. This will be very short, further details are in [2]. Let X be a *shape* in $\mathbb{R}^d$, i.e. a nice (finite, compact[1]) simplicial complex in $\mathbb{R}^d$. The Euler Characteristic ECT of X is defined as:

$$\begin{aligned}\mathrm{ECT}_X(v,t) &= \chi\big(X \cap H_{v,t}\big)\\ &= \chi\big(\{x \in X|\ x \cdot v \leq h\}\big),\end{aligned}$$

where χ is the Euler Characteristic, and $H_{v,t}$ is the half-space consisting of points less than height t in direction v. Here $v \in S^{d-1}$ and $h \in \mathbb{R}$.

We may also consider these problems with what is called *Persistent Homology Transform*, PHT. In PHT, we record the betti numbers instead of the Euler Characteristic.

Some facts about the ECT:

- The ECT is almost everywhere constant. We will call the locations where it is not constant *jumps*.
- The ECT can only jump at vertices of X. More precisely: if (v_0, t_0) is a jump point of the the ECT_X, then there exists a vertex p_0 of X that satisfies $p_0 \cdot v_0 = h_0$.
- The height functions partion $S^{d-1} \times \mathbb{R}$ into strata, and the ECT is constant away from the height functions.

[1] we may further assume X is supported in the unit ball.

Question 1

Given n points $X = \{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$, define an equivalence relation on linear functions $f : \mathbb{R}^d \to \mathbb{R}$ that are injective on X given by the order of the vertices x_i.

I a For a point cloud X, how many equivalence classes are there?

I b For a fixed d and n, which point configuration attains the maximum?

Question 2

An effective way to sort lists is the mergesort, which is a sort of divide and conquer algorithm.

In 2d, it is fairly easy to devise a divide and conquer algorithm that speeds the intersection checks exponentially over brute force.

II What is an effective way to check intersection of collection of triangles on S^2 (or S^d in general)

Question 3

Given a shape $X \subset \mathbb{R}^d$, define it's sinogram complexity $S(X)$ as the number of d-dimensional strata in $S^{d-1} \times \mathbb{R}$ its Euler characteristic transform.

III a What are the properties of $S(X)$?

III b Give a characterization of shapes of given sinogram complexity.

III c Give a fast to compute algorithm to approximate the sinogram complexity of a shape

III d Define a better notion of shape complexity that behaves better under e.g. set union

III e (Try any of the above with the PHT)

For example, $S(X) = 2$ is the set of convex bodies.

References

[1] L. Crawford, A. Monod, A. Chen, S. Mukherjee, and R. Rabadán, "Functional Data Analysis using a Topological Summary Statistic: the Smooth Euler Characteristic Transform," in *Journal of the American Statistical Association*, Volume 115, Number 531, pp. 1139–1150, 2020.

[2] J.Curry, S.Mukherjee and K.Turner: "How Many Directions Determine a Shape and other Sufficiency Results for Two Topological Transforms," on the arXiv as https://arxiv.org/abs/1805.09782.

[3] H.Kirveslahti, S.Mukherjee "Representing Fields without Correspondences: the Lifted Euler Characteristic Transform," J Appl. and Comput. Topology (2023). https://doi.org/10.1007/s41468-023-00133-w

[4] K. Turner, S. Mukherjee, and D. Boyer "Persistent Homology Transform for Modeling Shapes and Surfaces," *Information and Inference: A Journal of the IMA*, Volume 3, Number 4, 310–344, 2014.

[5] B. Wang, T.Sudijono, and H.Kirveslahti, T.Gao, D.M.Boyer, S.Mukherjee and L.Crawford. "A statistical pipeline for identifying physical features that differentiate classes of 3D shapes," *The Annals of Applied Statistics*, Volume 15, Issue 2, pp. 638-661, 2021.